

Stability and excitations of solitons in 2D Bose-Einstein condensates

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The small oscillations of solitons in 2D Bose-Einstein condensates are investigated by solving the Kadomtsev-Petviashvili equation which is valid when the velocity of the soliton approaches the speed of sound. We show that the soliton is stable and that the lowest excited states obey the same dispersion law as the one of the stable branch of excitations of a 1D gray soliton in a 2D condensate. The role of these states in thermodynamics is discussed.

Solitons in Bose-Einstein condensates (BECs) have recently attracted much attention and have been studied both theoretically and experimentally. The Gross-Pitaevskii (GP) equation admits solitonic solutions corresponding to a local density depletion (i.e., gray and dark solitons). They have been created in 3D BECs¹ and their decay into vortex rings has been observed². In 2D, an interesting type of soliton corresponds to a self-propelled vortex-antivortex pair that, when it moves at a velocity close to the Bogoliubov sound speed, c , takes the form of a rarefaction pulse³. The energy E and momentum \mathbf{P} were calculated in Ref.³, and it was found that the curve $E(P)$ is below the Bogoliubov sound line, approaching it in the low momentum limit. We investigate the excitation spectrum. Here we present our first results for solitons moving with velocity close to c , for which one can reduce the GP equation to the simpler Kadomtsev-Petviashvili (KP) equation^{3,4}.

Let us consider a 2D condensate with a soliton moving at velocity V in the x -direction. If the density at large distances is n_∞ , one can define the healing length $\xi = \hbar/(2mgn_\infty)^{1/2}$, where g is the mean-field coupling constant and m is the mass of the bosons. One can introduce the dimensionless variables $x \rightarrow \xi x$, $y \rightarrow \xi y$ and $t \rightarrow mt\xi^2/\hbar$, the normalized order parameter $\Psi \rightarrow \sqrt{n_\infty}\Psi$ and the velocity $U = m\xi V/\hbar = V/(c\sqrt{2})$ where $c = \hbar/(\sqrt{2}m\xi)$

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is the sound speed. In the frame moving with the soliton, the GP equation becomes³

$$2i\frac{\partial}{\partial t}\Psi = -\nabla^2\Psi + 2iU\frac{\partial}{\partial x}\Psi + (|\Psi|^2 - 1)\Psi. \quad (1)$$

The order parameter can be written as $\Psi = n^{1/2}e^{iS}$. In the limit of $V \rightarrow c$, one can introduce the small parameter $\varepsilon \equiv \sqrt{1 - 2U^2}$, so that $U \simeq 1/\sqrt{2} - 1/(2\sqrt{2})\varepsilon^2$. The density depletion associated with the soliton becomes shallow and one can expand the density and the phase as

$$n = 1 - \varepsilon^2 f + \dots \quad (2)$$

$$S = \varepsilon s + \dots \quad (3)$$

To the lowest order in ε , the GP equation gives $\partial_x s = -\varepsilon f/\sqrt{2}$ and the function f obeys the KP equation

$$\frac{\partial}{\partial \tilde{x}} \left(\frac{\partial f}{\partial \tilde{t}} + 6f \frac{\partial f}{\partial \tilde{x}} + \frac{\partial^3 f}{\partial \tilde{x}^3} \right) = \frac{\partial^2 f}{\partial \tilde{y}^2}, \quad (4)$$

where we have introduced the stretched variables $\tilde{x} = -\varepsilon x + \varepsilon^3 t/(2\sqrt{2})$, $\tilde{y} = \varepsilon^2 y/\sqrt{2}$ and $\tilde{t} = \varepsilon^3 t/(4\sqrt{2})$.

i) 1D gray soliton in 2D condensate.

By assuming the stationary solution of (4) to be independent of \tilde{y} , Eq. (4) reduces to the Korteweg-de Vries (KdV) equation and the solution is $f_0(\tilde{x} - 2\tilde{t}) = \text{sech}^2[(\tilde{x} - 2\tilde{t})/\sqrt{2}]$, corresponding to a gray soliton. The linear stability of this solution can be studied by taking $f(\tilde{x}, \tilde{y}, \tilde{t}) = f_0(\tilde{x} - 2\tilde{t}) + \psi(\tilde{x} - 2\tilde{t})e^{i(\tilde{k}\tilde{y} - \tilde{\omega}\tilde{t})}$ and linearizing Eq. (4) in $\psi(\tilde{x})$. One gets

$$\frac{d^4\psi}{d\tilde{x}^4} + 6\frac{d^2}{d\tilde{x}^2}(f_0\psi) - 2\frac{d^2\psi}{d\tilde{x}^2} + \tilde{k}^2\psi = i\tilde{\omega}\frac{d\psi}{d\tilde{x}}. \quad (5)$$

This equation was already solved by Zakharov⁵, by applying the inverse scattering method, and also by Alexander *et al.*⁶. The excitations which are localized along \tilde{x} and periodic along \tilde{y} have dispersion law $\tilde{\omega}^2 = (16\tilde{k}^2/3\sqrt{3})(\tilde{k} - \sqrt{3}/2)$. For a given ε , the eigenfrequencies ω in the original units of the GP equation (1) are given by $\omega = \varepsilon^3\tilde{\omega}/(4\sqrt{2})$, and the wavevector transforms as $k = \varepsilon^2\tilde{k}/\sqrt{2}$, so that

$$\omega^2 = \frac{k^2}{6} \left(\sqrt{\frac{8}{3}}k - \varepsilon^2 \right). \quad (6)$$

For $k > \varepsilon^2\sqrt{3/8}$ the frequency is real, while for $k < \varepsilon^2\sqrt{3/8}$ it is imaginary, causing the instability of the soliton against bending.⁴ In the limit $\varepsilon \rightarrow 0$, the

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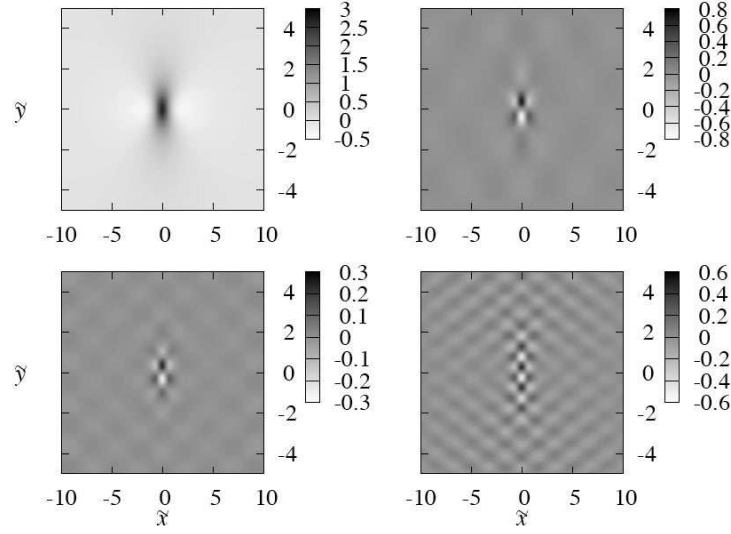


Fig. 1. Profile of the 2D soliton, Eq. (7), (upper left) and three examples of excited states with $\tilde{\omega} = 5.0$ (upper right), $\tilde{\omega} = 11.8$ (lower left), and $\tilde{\omega} = 24.4$ (lower right). For the excitations, the real part of the odd eigenfunctions is shown.

region of instability becomes vanishingly small and, for $k \gg \varepsilon^2$, the stable branch of excitations becomes $\omega = (2/27)^{1/4} k^{3/2}$. It is worth noticing that this dispersion exhibits the same power law as for capillary waves on the surface of a liquid.

ii) 2D soliton.

The solution of Eq. (4) corresponding to a 2D soliton is⁷

$$f_0(\tilde{x} - 2\tilde{t}, \tilde{y}) = 4 \frac{\frac{3}{2} + 2\tilde{y}^2 - (\tilde{x} - 2\tilde{t})^2}{[\frac{3}{2} + 2\tilde{y}^2 + (\tilde{x} - 2\tilde{t})^2]^2} . \quad (7)$$

This function is plotted in the upper left frame of Fig. 1. In the units of the GP equation (1), the density profile of the soliton is found by inserting (7) into (2). Its size along x and y is of the order of $1/\varepsilon$ and $1/\varepsilon^2$, respectively. To the first order in ε , the momentum P and the energy E per unit length are³

$$P = \frac{8\pi}{3} \sqrt{2} \hbar n_\infty \xi \varepsilon , \quad E = \frac{8\pi}{3m} \hbar^2 n_\infty \varepsilon = cP . \quad (8)$$

As it must be, the soliton has a sound-like dispersion in the limit where KP equation is valid, that is, when $P \rightarrow 0$.

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Now we consider fluctuations of the form $f(\tilde{x}, \tilde{y}, t) = f_0(\tilde{x} - 2\tilde{t}, \tilde{y}) + \psi(\tilde{x} - 2\tilde{t}, \tilde{y})e^{-i\tilde{\omega}\tilde{t}}$ and linearize Eq.(4) by ψ . Thus, $\psi(\tilde{x}, \tilde{y})$ satisfies

$$\frac{\partial^4 \psi}{\partial \tilde{x}^4} + 6 \frac{\partial^2}{\partial \tilde{x}^2} (f_0 \psi) - 2 \frac{\partial^2 \psi}{\partial \tilde{x}^2} - \frac{\partial^2 \psi}{\partial \tilde{y}^2} = i\tilde{\omega} \frac{\partial \psi}{\partial \tilde{x}} . \quad (9)$$

Looking for bound excited states (i.e., $\psi \rightarrow 0$ at large distances), it is convenient to integrate Eq. (9) twice along \tilde{x} and introduce the function $\phi(\tilde{x}, \tilde{y})$ such that $\psi = \frac{\partial^2 \phi}{\partial \tilde{x}^2}$. The equation for ϕ is

$$\frac{\partial^4 \phi}{\partial \tilde{x}^4} + 6f_0 \frac{\partial^2 \phi}{\partial \tilde{x}^2} - 2 \frac{\partial^2 \phi}{\partial \tilde{x}^2} - \frac{\partial^2 \phi}{\partial \tilde{y}^2} = i\tilde{\omega} \frac{\partial \phi}{\partial \tilde{x}} . \quad (10)$$

We numerically solve this equation in a square box of size L , by expanding ϕ as $\phi(\tilde{x}, \tilde{y}) = \sum_{\nu, \mu} \phi_{\nu\mu} \chi_{1,\nu}(\tilde{x}) \chi_{2,\mu}(\tilde{y})$ with $\chi_{1,\nu}(\tilde{x}) = L^{-1/2} e^{i2\pi\nu\tilde{x}/L}$ for $|\nu| \leq l$. Concerning $\chi_{2,\mu}(\tilde{y})$ we note that Eq. (9) is invariant for $\tilde{y} \rightarrow -\tilde{y}$, so that the function ϕ is either even or odd function of \tilde{y} . If it is even, then one can take $\chi_{2,\mu}(\tilde{y}) = (2/L)^{1/2} \cos(2\pi\mu\tilde{y}/L)$ for $1 \leq \mu \leq l$ and $\chi_{2,0}(\tilde{y}) = (1/L)^{1/2}$. If it is odd, one can take $\chi_{2,\mu}(\tilde{y}) = (2/L)^{1/2} \sin(2\pi\mu\tilde{y}/L)$ for $1 \leq \mu \leq l$. One thus obtain the following matrix equation

$$\left(-q_x^3 - 2q_x - q_y^2/q_x \right) \phi_{\nu\mu} + 6 \sum_{\nu', \mu'} q'_x M_{\nu\mu, \nu'\mu'} \phi_{\nu'\mu'} = \tilde{\omega} \phi_{\nu\mu} , \quad (11)$$

with $q_x = 2\pi\nu/L$, $q_y = 2\pi\mu/L$ and

$$M_{\nu\mu, \nu'\mu'} = \int_{-L/2}^{L/2} d\tilde{x} d\tilde{y} \chi_{1,\nu}^* \chi_{2,\mu}^* f_0 \chi_{1,\nu'} \chi_{2,\mu'} . \quad (12)$$

The size of the matrix is $N \times N$, where $N = 2l(l+1)$ for ϕ even, and $N = 2l^2$ for ϕ odd. Typical values in our calculations are $l = 70$ and $L = 60$.

It is found that all the eigenvalues are real. This is consistent with the results of Refs.^{8,9} and shows the linear stability of the 2D soliton. One can expect this result also from pure kinematic considerations. An excited state is unstable if it can decay into several phonons. Energy and momentum must be conserved in this process. It was proved by Iordanskii and one of the authors¹⁰ that for a Bogoliubov-like spectrum the conservation laws can be satisfied only if the dispersion law $E(P)$ is above the sound line. Since the dispersion law of the 2D soliton is below the sound line, the soliton is stable.

We find stable localized modes with positive eigenvalues. Examples are shown in Fig. 1 for $\tilde{\omega} = 5.0, 11.8$ and 24.4 . The function ψ oscillates along \tilde{x} and \tilde{y} , with pronounced maxima in the soliton region. Since the density

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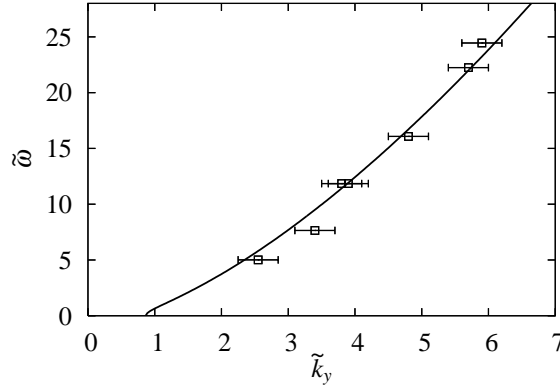


Fig. 2. Excitation spectrum of the 2D soliton. Points are obtained from Eq. (10). The solid line is the dispersion law of the stable branch of excitations of a 1D gray soliton, Eq. (6).

profile of the soliton decays algebraically, $\psi(\tilde{x}, \tilde{y})$ decays rather slowly. It is found that the real and imaginary part of ψ are either even or odd functions of \tilde{x} and each eigenvalue is degenerate, corresponding to even and odd functions of \tilde{y} .

By looking at the oscillations of the eigenvectors ψ in the soliton region, one can estimate the transverse and axial wavevectors, \tilde{k}_x and \tilde{k}_y respectively. In Fig. 2 we plot the calculated eigenfrequencies $\tilde{\omega}$ as a function of \tilde{k}_y for the lowest bound states (points with error bars, where the error bars are of the order of the inverse of the axial size of the soliton). In the same figure we plot the dispersion law (6) of the stable branch of excitations of a 1D gray soliton. The spectrum is very similar. The 2D soliton have a discrete spectrum of bound states, due to its finite size, and all wavevectors are above the threshold for the instability of the 1D soliton.

Notice that the soliton oscillations can be subject to damping by emission of sound waves. This means that the energy levels of Fig. 2 have to be considered as resonance states.¹¹ However, there are reasons to believe that this damping is small and can be neglected both in the calculation of the energy levels and in the thermodynamic considerations below. We plan to investigate this damping in the future.

As it was noticed in Ref.³, this 2D soliton can contribute to the specific heat C of the 2D Bose gas. Indeed, if one applied the usual Bose statistics to the soliton branch, due to its sound-like dispersion (8), its contribution would be $C \propto T^2$ as for phonons. The presence of the excited states of the soliton can change the situation. The inclusion of these excitations is an

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open problem; here we just want to provide an argument which illustrates its nontriviality.

One can take into account the excitations of solitons by assuming that the solitonic branch of the spectrum depends on two quantum numbers: \mathbf{P} and k . For example, within the limits of applicability of the dispersion law (6), one can write

$$E(\mathbf{P}, k) = c|\mathbf{P}| + \beta k^{3/2}. \quad (13)$$

One has to accept the hypothesis that the number of states in the interval d^2P is, as usual, $Sd^2P/(2\pi\hbar)^2$, where S is the sample area. On the contrary, the quantization rule for k is fixed by the length of the soliton in the y direction, R_y , which behaves as $R_y \propto \xi/\varepsilon^2$. Correspondingly, the number of states in the interval dk is $\propto R_y dk/(2\pi)^3 \propto (\xi/\varepsilon^2)dk$. The energy of the gas is thus $\mathcal{E} \propto \int d^2P dk \varepsilon^{-2} E(\mathbf{P}, k) [\exp(E(\mathbf{P}, k)/T) - 1]^{-1}$. A simple calculation shows that the relevant values are $E \propto \varepsilon \propto T$, $P_i \propto T$ with $i = x, y$, and $k \propto T^{2/3}$. This gives the specific heat $C \propto T^{2/3}$, which exceeds the phonon contribution at low T . Of course, this result cannot be considered as a rigorous one, in particular because solitons occupy an area $\propto 1/\varepsilon^3$ and they can overlap at low temperature. Nevertheless it shows that the problem of the low-temperature specific heat of the 2D Bose gas deserves both theoretical and experimental investigation. A direct diagonalization of the Hamiltonian or Monte Carlo simulations at finite temperature could be suitable tools to test our semi-quantitative prediction.

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